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Hypersonic, turbulent viscous interaction past an expansion corner

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The University of Texas at Arlington, 1992
HYPersonic, TURBulent Viscous Interaction

Past an Expansion Corner

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HYPersonic, TURBulent VisCOuS INTERAcTion
Past an EXPAnsion CORNER

by

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Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

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ABSTRACT

HYPERSONIC, TURBULENT VISCOUS INTERACTION
PAST AN EXPANSION CORNER

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Behzad Bigdeli, M.S.
The University of Texas at Arlington, 1992

Supervising Professor: Frank K. Lu

A study was made of turbulent viscous interaction on a flat plate at zero incidence, a compression corner and an expansion corner at hypersonic speeds. By assuming a pressure law, the boundary layer properties of the flow were obtained by simultaneous solution of a displacement thickness relationship and a coupling equation relating the effects of incidence and displacement thickness to the effective body shape. The tangent-wedge rule was employed to predict the pressure distribution over the flat plate and the compression corner, while a new pressure law approximation for expansion corner was proposed. The effect of wall temperature ratio on the displacement thickness ratio was studied and a new representative value of $n$ in the power law assumption was obtained. The method was extended to turbulent supersonic flows past expansion corners. The results were compared with the experimental data for both hypersonic and supersonic cases.
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NOMENCLATURE

A \, \gamma, \text{ constant}

a, m \, P = a x^m, \text{ constants}

B \, \frac{\gamma(\gamma + 1)}{2}, \text{ constant}

b, s \, P = b \exp(sx), \text{ constants}

C_\infty \, \text{constant in linear viscosity-temperature law}

K \, M_\infty \frac{dy_e}{dx}, \text{ flow parameter}

L \, \text{length of the flat plate to the hinge line, cm}

l \, \text{scale variable}

M \, \text{Mach number}

n \, \text{power law variable}

P \, \text{pressure ratio}

Pr \, \text{Prandtl number}

Re \, \text{unit Reynolds number}

r \, \text{recovery factor}

T \, \text{temperature, K}

u \, \text{velocity, m/s}

X \, \text{normalized form of the physical coordinate x}

x \, \text{distance from the leading edge along the plate, cm}

Y \, \text{Howarth's transformation variable}

y \, \text{distance normal to the surface, cm}

y_e \, \text{effective body shape}

y_w \, \text{geometric body shape}
\( \alpha \) corner angle, degree
\( \gamma \) ratio of specific heats
\( \Delta \) transformed boundary layer thickness
\( \delta \) boundary layer thickness, cm
\( \delta^* \) displacement thickness, cm
\( \eta \) \((5.75 - 1.625 \frac{T_w}{T_o})\), empirical variable
\( \kappa \) \((6 - 1.3 \frac{T_w}{T_o})\), empirical variable
\( \mu \) absolute viscosity coefficient
\( \xi \) dummy variable of integration
\( \rho \) density
\( \bar{X}_s \) strong viscous interaction parameter
\( \bar{X}_w \) weak viscous interaction parameter

Subscripts
\( b \) bluntness
\( e \) condition at edge of boundary layer
\( I \) inviscid
\( o \) stagnation condition
\( r \) recovery condition
\( t \) turbulent
\( w \) wall condition
\( \infty \) freestream condition
1. INTRODUCTION

In recent years increasing interest in the development of flight vehicles at extremely high velocities has motivated a large number of research studies. At hypersonic speeds where the flight velocity is far greater than the speed of sound, the characteristics of the flow can be drastically different than those at supersonic speeds. These hypersonic features have brought a set of new fluid dynamic problems into prominence such as rarefied gases, dissociation effects and viscous interactions.

Viscous interaction was identified in view of the pressure measurements near the leading edge of a wedge for the first time by Becker [1]. He found that the actual pressure along the wedge is considerably higher than the values of the inviscid pressure obtained from the oblique shock theory. Viscous interaction or pressure interaction [2] can be viewed as the mutual interaction between the external inviscid flow field and the boundary layer around a body. The boundary layer grows for example on a flat plate as $M^2/\sqrt{\text{Re}_\delta}$ [3]. Therefore, relatively speaking, in subsonic and most supersonic flows where the Reynolds number is much larger than the square of the Mach number, the effect of boundary layer growth in changing the effective body shape and the actual pressure distribution is very small and can be ignored in general. In hypersonic flight, viscous interaction becomes significant. This is due to the fact that hypersonic flow contains a large amount of kinetic energy some of which will be dissipated once the flow is slowed
by viscous effects within the boundary layer. The kinetic energy is converted into the internal energy of the gas and the process is called the viscous dissipation. The effects of compression and energy dissipation impart a considerable temperature rise. This temperature rise causes the coefficient of viscosity to increase. Also, from the equation of state it can be seen that the density decreases by increasing temperature since pressure is approximately constant in normal direction. The reduced density requires a larger boundary layer thickness to maintain the same mass flow through the boundary layer. Therefore, the boundary layer grows rapidly. This thick boundary layer distorts the external inviscid flow field severely which in turn modifies the boundary layer growth. Therefore, the prediction of such mutual interactions is necessary to correctly design a shape which satisfies certain performance requirements. For certain flows, this task is accomplished by including the effect of the boundary layer displacement thickness in the effective body shape and calculating the corresponding pressure distribution.

There are several examples of viscous interaction which are of interest in both laminar and turbulent boundary layers. Some of those that have been studied analytically include hypersonic flow near the leading edge of a sharp flat plate, unseparated shock boundary layer interaction in turbulent flow, turbulent interaction on curved surfaces and the turbulent flow at a wedge compression or expansion corner. For any shape with a sharp edge, a rapid change of boundary layer growth occurs in the regions of strong favorable or adverse pressure gradient whether the flow is laminar or turbulent. The strong pressure gradient is usually caused by changes of the surface slope, for example, at the hinge line of a corner.
Hayes and Probstein [4] epitomized viscous interaction problem in their book. Cheng et al. [5] studied the laminar hypersonic viscous interaction. They were able to demonstrate the mutual effects of the surface incidence and the boundary layer growth. Cheng et al. [5] adopted Lees' [6] similarity theory for local flat plates at hypersonic boundary layers and they used the Busemann [7] centrifugal correction to the Newtonian theory. They found parameters governing the flow over any shape with a sharp edge. These parameters reflect the effect of boundary layer growth, incidence, strong viscous interaction and bluntness. Cheng et al. applied the analysis to a set of flat plates at different incidence in laminar flow only. Moreover, the choice of the Newton-Busemann law [7] is appropriate only for strong viscous interaction regions since it does not tend to the correct downstream value and as shown by Mohammadian [8] and Cheng and Kirsch [9], it leads to physically unrealistic, highly-oscillatory solutions for some curved surfaces.


Barnes and Tang [12] investigated mass injection and formulated the
strong and weak interaction parameters in turbulent flow. Stollery and Bates [13] provided a similar theory to the laminar case for the turbulent flow. They showed that the key to the method is to express the turbulent boundary layer growth in terms of an initially unknown pressure distribution $P(x)$. They also pointed out that the turbulent viscous interaction on curved surfaces and shock-wave boundary layer interactions are important. Stollery and Bates [13] employed the momentum-integral method of Spence [14] for a more detailed prediction and made comparisons with experimental data obtained by Coleman [15] and Elfstrom [16].

The aim of the present study is to partially reproduce the results obtained by Stollery and Bates [13] for turbulent hypersonic viscous interaction and to extend the analysis to an expansion corner. The possibility of suggesting a new pressure law approximation in place of the tangent-wedge rule approximation for the expansion corner will be investigated. Also, the method is extended to supersonic flow past expansion corners. The results will be compared with the experimental data obtained by Bloy [17] and Lu and Chung [18] for the hypersonic case and those obtained by Goldfeld [19] for the supersonic case.
2. VISCOUS INTERACTION

2.1 Problem in General

Hypersonic, viscous flow past an arbitrary body is affected by three different effects [20]. These effects are the incidence or geometric body shape \( y_w \), the boundary layer growth or displacement thickness \( \delta^* \) and the bluntness of the shape or entropy layer \( y_b \). The entropy layer is defined as a region of strong entropy gradients which is created in the nose region of a blunt nose body. The boundary layer grows inside the entropy layer and is affected by it. This interaction is also called vorticity interaction [3].

At hypersonic speeds the boundary layer is thick. Therefore, the effective body shape \( y_e \) can no longer be represented by the surface of the body. Instead, the effective body shape should be considered as the surface of the body plus the displacement thickness. Figure 1 illustrates the geometric body shape, displacement thickness, entropy layer and the effective body shape and their notations for a blunt nosed body.

2.2 Bodies With Sharp Leading Edges

In this study the emphasis is on bodies with a sharp leading edge and bluntness is ignored. In the other words, the effective body shape is the same as the entropy layer, i.e. \( y_e = y_b \). Hypersonic flow past bodies with sharp leading edges can be represented by three equations. These equations are the boundary
Figure 1: Surface Features of a Blunt Nosed Body.
layer equation, the inviscid flow equation and a coupling equation. Mathematically these equations can be written functionally as

\[ \delta^* = f_1(P) \]  \hspace{1cm} (1)

\[ P = f_2(y_e) \]  \hspace{1cm} (2)

\[ y_e = f_3(\delta^*) \]  \hspace{1cm} (3)

where \( P \) is an initially unknown pressure distribution \( \frac{P_e(x)}{P_\infty} \) at position \( x \). The pressure is assumed constant across the boundary layer so that \( P_e = P_w \), but due to viscous interaction the pressure at the outer edge of the boundary layer \( P_e \) is not the same as the free stream pressure \( P_\infty \). From this very general statement of the problem one can conclude that a complete solution to the flow past a given shape can be obtained once the boundary layer growth, external pressure distribution and the effective body shape are known. Many different choices for these three equations are possible. Rather simple forms of these equations will be employed in order to obtain faster solution.

2.2.1 Displacement Thickness

A general expression for displacement thickness for turbulent flows was obtained by Stollery and Bates [13] based on the momentum-integral relation. They adopted the analysis given by Spence [14]. The final result is:
\[
\frac{M\delta^*}{x} = 0.051 \frac{1+1.3 \frac{T_w}{T_o}}{(1+2.5 \frac{T_w}{T_o})^{3/2}} \left( \frac{M_{\infty}^2 C_{\infty}}{Re_z} \right)^{1/6} \left( \int_0^x \frac{P^\eta d\zeta}{x} \right)^{1/6} \tag{4a}
\]

where

\[
\eta = \frac{1}{3} \left( 5.75 - 1.625 \frac{T_w}{T_o} \right) \tag{4b}
\]

\[
\kappa = \frac{1}{4} \left( 6 - 1.3 \frac{T_w}{T_o} \right) \tag{4c}
\]

\(M_{\infty}\) is the freestream Mach number, \(T_w\) is the wall temperature, \(T_o\) is the stagnation temperature, \(C_{\infty}\) is the constant of proportionality in the linear viscosity-temperature relationship, \(Re_z\) is the Reynolds number based on the distance \(x\) from the leading edge, \(P\) is an initially unknown pressure distribution and \(\zeta\) is the dummy variable of integration in \(x\). Equation \(4a\) describes the growth of the boundary layer displacement thickness \(\delta^*\) in an unknown pressure gradient for any given Mach number and wall temperature ratio. If \(P\) is assumed constant in equation \(4a\), it can be shown that:

\[
\frac{\delta^*}{x} = 0.051 \frac{1+1.3 \frac{T_w}{T_o}}{(1+2.5 \frac{T_w}{T_o})^{3/2}} \left( \frac{M_{\infty}^4 C_{\infty}}{Re_z P} \right)^{1/6} \tag{4d}
\]

It is easy to deduce from equation \(4d\) that for given wall temperature conditions,

\[
\frac{\delta^*}{x} \sim \left( \frac{M_{\infty}^4 C_{\infty}}{Re_z P} \right)^{1/6} \tag{4e}
\]
Equation (4e) shows that the displacement thickness increases with increasing Mach number, and decreases in an adverse pressure gradient. Further, equation (4d) reveals that the effect of wall temperature on the displacement thickness is very weak, with the displacement thickness $\delta^*$ increasing slightly with increasing wall temperature ratio $\frac{T_w}{T_o}$. Stollery and Bates [13] showed that equation (4e) is only appropriate for flows over flat plates whereas for general surfaces the less approximate relationship, equation (4a), should be used.

Spence [14] showed that the analysis of a compressible turbulent boundary layer, as in the laminar case, can be greatly simplified by replacing the physical coordinate $y$ with a similarity variable $Y$. He adopted Howarth's transformation [23] or

$$ Y = \int_0^y \frac{\rho}{\rho_e} dy $$

where $\rho$ is the density at height $y$ above the surface and the subscript $e$ denotes the condition at the outer edge of the boundary layer. First, he showed that for a laminar boundary layer on a flat plate, if the product of the density and the coefficient of viscosity is constant the similarity transformation allows the velocity profile to be presented in a form which accounts for the effect of compressibility and shows no explicit dependence on Mach number and Prandtl number, i.e.

$$ \frac{u}{u_e} = f\left(\frac{Y}{\Delta}\right) $$

where $f$ is a universal function and $\Delta$ is a transformed boundary layer thickness.
\[ \Delta = \int_0^\delta \frac{\rho}{\rho_c} \, dy \quad \text{(6b)} \]

For turbulent flows Spence [14] showed that if the velocity profile obeys a power law of the form

\[ \frac{u}{u_c} = \left( \frac{Y}{\Delta} \right)^{\frac{1}{n}} \quad \text{(6c)} \]

the results compare well with the experimental data. Stollery [22] extended Spence's analysis [14] to obtain a simple expression for the displacement thickness ratio \( \frac{\delta^*}{\delta} \). This analysis is as follows. By definition:

\[ \delta^* = \int_0^\delta \left( 1 - \frac{\rho u}{\rho_c u_c} \right) \, dy \quad \text{(7a)} \]

Using the assumptions given in equations (6a) and (6b), equation (7a) can be written as:

\[ \delta^* = \delta - \int_0^\Delta f\left( \frac{Y}{\Delta} \right) \, dY = \delta - \Delta J \quad \text{(7b)} \]

where

\[ J = \int_0^1 f\left( \frac{Y}{\Delta} \right) d\left( \frac{Y}{\Delta} \right) \quad \text{(7c)} \]

Using equation (6c) the integral J, equation (7c) can be evaluated to yield
\[ J = \frac{n}{n + 1} \]  

The boundary layer thickness \( \delta \) can be written as:

\[ \delta = \int_0^\delta dy = \int_0^\Delta \frac{\rho_e}{\rho} \, dY \]  

Applying the equation of state to equation (8a) gives:

\[ \delta = \int_0^\Delta \frac{T}{T_e} \, dY \]  

Assume that the linear Crocco [23] temperature-velocity relationship holds in the form

\[ \frac{T}{T_e} = \frac{T_w}{T_e} + \left( \frac{T_r - T_w}{T_e} \right) \frac{u}{u_e} - \left( \frac{T_r - T_e}{T_e} \right) \left( \frac{u}{u_e} \right)^2 \]  

In this relationship \( T_r \) is the recovery temperature, \( T_e \) is the temperature at the outer edge of the boundary layer and \( T_w \) is the wall or the surface temperature. The recovery temperature is defined as:

\[ T_r = T_\infty \left( 1 + r \frac{\gamma - 1}{2} M_\infty^2 \right) \]  

where \( r \) is the recovery factor and is a function of the Prandtl number. White [24] states the recovery factor for laminar and turbulent flows based on the experimental results as follows:
\[ r = Pr^{\frac{1}{2}} \]  
\[ r \approx Pr^{\frac{1}{3}} \]

where \( Pr \) is the turbulent Prandtl number. In this analysis a turbulent Prandtl number of 0.89 is assumed which corresponds to a recovery factor of 0.9. Using equation (7b) the displacement thickness ratio can be shown to be:

\[
\frac{\delta^*}{\delta} = \frac{\delta - \Delta J}{\delta} = 1 - \frac{\Delta J}{\delta} \tag{10a}
\]

Substituting the expressions obtained in equations (7d), (8b) and (9a) in equation (10a) and simplifying (Appendix A) yields the following expression for the displacement thickness ratio:

\[
\frac{\delta^*}{\delta} = 1 - \frac{n+2}{(T_r/T_e) + \left(\frac{n+2}{n}\right)\left(T_w/T_e\right) + (n+1)} \tag{10b}
\]

Equation (10b) explicitly shows the dependence of the displacement thickness ratio of a hypersonic turbulent boundary layer on the wall temperature ratio. The value of \( n \) can vary from 6 to 10, depending on the Reynolds number. The higher the value of the Reynolds number, the fuller the velocity profile and the larger the value of \( n \). The effect of wall temperature conditions on the displacement thickness ratio is illustrated in figure 2 which plots equation (10b) for four different cases using a representative value of \( n \) equal to 8.5. It can be seen that the displacement thickness ratio under the adiabatic conditions (\( T_w = T_e \)) is slightly higher than the values obtained under cold wall conditions (\( T_w = 0 \)).
Figure 2: Displacement Thickness Ratio, $n = 8.5$. 
Also, a comparison of the theoretical values with experimental data of Hopkins et al. [25] for $\frac{T_w}{T_r} = 0.3$ and 0.5 shows that the trend with $\frac{T_w}{T_r}$ is reflected accurately and the theoretical and the experimental data are in reasonably good agreement.

2.2.2 Pressure Laws

The change in pressure can be considered to be the most important consequence of viscous interaction. Therefore, it is important in any analysis dealing with hypersonic viscous interactions to be able to describe the pressure. For a wide range of conditions applying viscous-interaction theory, the tangent-wedge rule has been found to be adequate. Sullivan [10] and Stollery [11] showed that

\[ P = 1 + \gamma K^2 \left( \frac{\gamma + 1}{4} + \left( \frac{\gamma + 1}{4} \right)^2 + \frac{1}{K^2} \right)^{\frac{1}{2}} \]  
\[ \text{(11a)} \]

where

\[ K = M_\infty \frac{dy_e}{dx} \]  
\[ \text{(11b)} \]

The tangent-wedge rule describes the static pressure $P$ as a function of the effective body shape $y_e$. The pressure distribution calculated from the tangent-wedge rule must tend asymptotically to the inviscid two-dimensional value. The final inviscid two-dimensional value of pressure is given by:

\[ P_I = 1 + \gamma (M_\infty \alpha)^2 \left( \frac{\gamma + 1}{4} \pm \left( \frac{\gamma + 1}{4} \right)^2 + \frac{1}{(M_\infty \alpha)^2} \right)^{\frac{1}{2}} \]  
\[ \text{(11c)} \]
where \( \alpha \) is the corner angle in radian and \( M_\infty \) is the freestream Mach number. The plus and minus signs correspond to the compression and expansion cases respectively. Either one of the equations (11c) can be used to obtain the final inviscid value of the pressure for a flat plate at zero incidence since both equations for \( K = 0 \) yield \( P = 1 \). Equation (11c) with the minus sign shows the range of validity of this pressure law for the expansive cases. Since a negative value for pressure is not physically possible the lowest possible value of \( P \) is zero which corresponds to a value of \( M_\alpha \) equal to \(-\left(\frac{2}{\gamma(\gamma-1)}\right)^{\frac{1}{2}}\). For air with \( \gamma = 1.4 \), this corresponds to a value of \( M_\alpha = -1.89 \). Therefore, the tangent-wedge rule for expansion corner flows is limited and it is necessary to obtain a more general pressure law which can cover a broader range of \( M_\alpha \).

2.2.3 Effective Body Shape

The effective body shape \( y_e \) can be expressed as

\[
y_e = y_w + \delta^* \tag{12a}
\]

where \( y_e \) can be differentiated with respect to \( x \) to yield

\[
\frac{dy_e}{dx} = \frac{dy_w}{dx} + \frac{d\delta^*}{dx} \tag{12b}
\]

This is only an approximate relationship. The exact relationship for this coupling equation is given by Lees [6] and Reeves [26], which is repeated as follows:
\[
\frac{dy_e}{dx} = \frac{dy_u}{dx} + \frac{d\delta^*}{dx} - (\delta - \delta^*) \frac{d}{dx} \ln(\rho_e u_e)
\]  \hspace{1cm} (12c)

For hypersonic boundary layers, as the Mach number tends to infinity, the displacement thickness becomes the same as the boundary layer thickness. Therefore, the last term in equation (12c) becomes small and can be neglected.
3. VISCOS INTERACTION REGIONS AND PARAMETERS

Viscous interactions can be classified into two distinct regions of strong and weak interactions. Figure 3 shows a schematic of strong and weak viscous interaction regions for flow past a flat plate at zero incidence. These regions are described in the following sections.

3.1 Strong Viscous Interaction Region

The strong viscous interaction region can be defined as a region in which the mutual interaction of the boundary layer and the inviscid flow are strong in which \( \frac{d\bar{y}}{dx} \) and \( \frac{d\bar{x}}{dx} \) are large. Therefore, the incoming freestream flow is significantly disturbed which in turn causes substantial changes in the boundary layer properties. Compared to the strong effect of the boundary layer displacement thickness, the effect of incidence in this region is small. The strong viscous interactions occur very near the nose and it is very unlike that these interactions are turbulent.

3.2 Weak Viscous Interaction Region

In contrast to the strong viscous interaction region, the rate of growth of the displacement thickness \( \frac{d\bar{y}}{dx} \) and the effective body shape \( \frac{d\bar{y}}{dx} \) in the weak viscous interaction region are small. Streamlines are slightly deflected into the incoming freestream flow. Hence, freestream distortion is negligible which results
Figure 3: Flat Plate Viscous Interaction Regions.
in weak changes in the boundary layer properties. In this region the dominant factor affecting the flow is the incidence.

3.3 Strong and Weak Viscous Interaction Parameters

It is necessary to analyze any hypersonic viscous flow problem for the possibility of existence of viscous interactions in order to make sure whether these effects need to be included. Hence, it is important to identify parameters which govern these effects. Tang and Engh [27] suggested a modified form of the laminar Lees-Probstein hypersonic interaction parameter that can be used to express both strong and weak viscous interactions in turbulent flow. This modified form is

\[ \bar{X} = M_\infty^3 \left( \frac{C_\infty}{Re_x} \right)^{\frac{3}{2}} \]  
\[ (13) \]

Barnes and Tang [12] showed that equation (13) with a suitable interaction equation overpredicts the induced pressures at high Mach numbers. Therefore, they formulated the strong and weak viscous interaction parameters for turbulent flow. These parameters are

\[ \bar{X}_S = \left( \frac{M_\infty^3 C_\infty}{Re_x} \right)^{\frac{3}{2}} \]  
\[ (14) \]

for the strong turbulent viscous interaction and

\[ \bar{X}_W = \left( \frac{M_\infty^3 C_\infty}{Re_x} \right)^{\frac{1}{2}} \]  
\[ (15) \]
for the weak turbulent viscous interaction. It is worth noting that viscous interactions are not always the dominant factor in all hypersonic flows. If $\bar{x}_w \ll 1$, then viscous interaction is unimportant.
4. APPLICATION AND DISCUSSION

4.1 Flat Plate at Zero Incidence

A review of the Stollery and Bates [13] work shows that the pressure distribution over a flat plate in hypersonic, turbulent viscous flow can be predicted using equations (4a), (11a) and (12a). The geometric body shape for a flat plate at zero incidence is \( y_w(x) = 0 \). Therefore, in equation (12a) the effective body shape, becomes

\[
y_e(x) = \delta^*(x) \tag{16a}
\]

which can be differentiated with respect to \( x \) to yield

\[
\frac{dy_e(x)}{dx} = \frac{d\delta^*(x)}{dx} \tag{16b}
\]

Therefore, parameter \( K \) given by equation (11b) can be written as

\[
K = M_\infty \frac{dy_e(x)}{dx} = M_\infty \frac{d\delta^*(x)}{dx} \tag{17}
\]

This equation can be substituted into the tangent-wedge approximation, equation (11a), to yield
\[ P = 1 + \gamma \left( M_\infty \frac{d\delta^*}{dx} \right)^2 \left( \frac{\gamma + 1}{4} + \left( \frac{\gamma + 1}{4} \right)^2 + \frac{1}{\left( M_\infty \frac{d\delta^*}{dx} \right)^2} \right)^{\frac{1}{2}} \]  

(18)

An explicit relationship for the displacement thickness \( \delta^*(x) \) can be obtained by multiplying both sides of equation (4a) by \( \frac{x}{M_\infty} \) as follows:

\[ \delta^* = a_1 \left( \frac{M_\infty}{Re} \right)^\frac{1}{6} \left( \frac{\int_0^x \frac{\tau_{\infty}}{\rho} d\xi}{x} \right)^{\frac{4}{5}} \]  

(19a)

where

\[ a_1 = 0.051 \frac{1+1.3 \frac{T_\infty}{T_\omega}}{\left( 1 + 2.5 \frac{T_\infty}{T_\omega} \right)^{\frac{3}{5}}} \]  

(19b)

The definition of Reynolds number can be used to simplify equation (19a) to

\[ \delta^* = a_1 \left( \frac{M_\infty}{Re} \right)^\frac{1}{6} \left( \frac{\int_0^x \frac{\tau_{\infty}}{\rho} d\xi}{x} \right)^{\frac{4}{5}} \]  

(19c)

The boundary layer displacement thickness \( \delta^*(x) \) can be calculated using equation (19c) if \( P \) is known [22]. \( P \) is an initially unknown pressure distribution which can be specified knowing the behavior of the displacement thickness. Since two distinct regions of strong and weak viscous interactions are already defined it seems reasonable to assume two different forms for the pressure distribution \( P \) corresponding to these regions. This assumption leads to two sets of solutions for the displacement thickness and the pressure distribution. The first is a strong
solution for the region close to the leading edge of the sharp flat plate and the second is a weak solution further back on the flat plate.

4.1.1 Strong Solution

If transition occurs in a region very close to the leading edge of a flat plate there will be a strong turbulent viscous interaction region. In this region the pressure $P$ can be assumed to take the form [22]

$$P = ax^m$$

(20)

The pressure distribution $P$ in equation (19c) can be replaced by equation (20) and the obtained relationship can be integrated to yield

$$\delta^* = a_1 \left( \frac{M_{\infty}^4 C_{\infty}}{Re_x} \right)^{\frac{1}{5}} a \left( \frac{\frac{4}{3} \eta - \kappa}{(m \eta + 1)^{\frac{3}{5}}} \right) x^{\left( \frac{4}{5} + \frac{4}{3} m \eta - m \kappa \right)}$$

(21a)

This equation can be differentiated with respect to $x$ to yield

$$\frac{d \delta^*(x)}{dx} = \left( \frac{4}{5} + \frac{4}{3} m \eta - m \kappa \right) a_1 \left( \frac{M_{\infty}^4 C_{\infty}}{Re_x} \right)^{\frac{1}{5}} a \left( \frac{\frac{4}{3} \eta - \kappa}{(m \eta + 1)^{\frac{3}{5}}} \right) x^{\left( -\frac{1}{5} + \frac{4}{3} m \eta - m \kappa \right)}$$

(21b)

Since the rate of change of the displacement thickness is large in the strong viscous interaction region, $K$ is also large and the tangent-wedge approximation of equation (11c) can be written as
\[
P = \gamma \frac{\gamma + 1}{2} K^2 + \left(\frac{\gamma + 3}{\gamma + 1}\right) - \frac{8\gamma}{(\gamma + 1)^3} \frac{1}{K^2} + ... \\
\approx \gamma \frac{\gamma + 1}{2} K^2
\] (22a, 22b)

Equation (21b) can be substituted into equation (22b) and simplifying yields

\[
P = \gamma \frac{\gamma + 1}{2} \left(\frac{4}{5} + \frac{4}{5} m \eta - m \kappa\right)^2 a_1^2 \left(\frac{M_{\infty}^{14} C_{\infty}}{\text{Re}}\right)^{\frac{4}{5}} \left(\frac{\frac{4}{5} - \kappa}{(m \eta + 1)^{\frac{4}{5}}}\right)^2 \chi^{\left(-\frac{2}{3} + \frac{2}{3} m \eta - 2m \kappa\right)}
\] (23)

Comparing equations (20) and (23) reveals that

\[
m = \frac{8}{3} m \eta - 2m \kappa - \frac{2}{5}
\] (24a)

This equation can be solved for \(m\) and the result is

\[
m = \frac{2}{8\eta - 10\kappa - 5}
\] (24b)

where \(\eta\) and \(\kappa\) are empirical variables which depend on the wall temperature ratio and are given in equations (4b) and (4c) respectively. It is interesting to note that the value of \(m\) is independent of the wall temperature ratio. For example, under cold-wall conditions, \(\eta\) and \(\kappa\) are equal to \(\frac{5.75}{7}\) and \(\frac{6}{7}\) respectively which correspond \(m = -\frac{2}{7}\) while under adiabatic wall conditions \(\eta\) and \(\kappa\) are equal to \(\frac{4.125}{7}\) and \(\frac{4.7}{7}\) respectively but the value of \(m\) remains the same. Assuming that \(a = 1\) and using the obtained value of \(m = -\frac{2}{7}\), equation (21b) leads to
\frac{d \delta^*(x)}{dx} = \left( \frac{4}{35} - \frac{8}{7} \eta + \frac{3}{5} \kappa \right) a_1 \left( \frac{M_{\infty}^4 C_{\infty}}{Re} \right)^{\frac{1}{6}} (-\frac{2}{7} \eta + 1)^{-\frac{3}{5}} x^{(\frac{3}{5} - \frac{5}{35} \eta + \frac{3}{5})} \tag{25}

Therefore, the strong solution for hypersonic, turbulent flow close to the leading edge of a sharp flat plate at zero incidence for a given set of initial conditions can be obtained by solving equations (18) and (25) simultaneously. The solution includes the rate of growth of the displacement thickness and the pressure distribution.

4.1.2 Weak Solution

Further back on the flat plate in the weak viscous interaction region the pressure on a sharp flat plate can be predicted using the tangent-wedge approximation for pressure at the edge of the effective surface. The tangent-wedge approximation for this region can be written as

\[ P = 1 + \gamma K + \gamma \frac{\gamma + 1}{4} K^2 + \ldots \tag{26a} \]

\[ \approx 1 + \gamma K \tag{26b} \]

Further, since $K \ll 1$, $P$ can be assumed to be equal to unity. Therefore, $P$ in the general expression obtained for the displacement thickness, equation (19c), can be replaced with unity and the obtained expression can be integrated to yield the following expression:

\[ \delta^* = a_1 \left( \frac{M_{\infty}^4 C_{\infty}}{Re} \right)^{\frac{1}{6}} x^{\frac{4}{5}} \tag{27a} \]
This expression is valid for predicting the growth of the displacement thickness over a sharp flat plate in the weak viscous interaction region. Equation (27a) shows that the displacement thickness varies with distance $x$ from the leading edge as

$$
\delta^* \propto x^{\frac{4}{3}}
$$

(27b)

This can be expressed in terms of the effective body shape as

$$
y_e \propto x^{\frac{4}{3}}
$$

(28)

which compares well with conventional results for the boundary-layer growth over a sharp flat plate. Also, this result is significantly different from the conventional laminar case [3] where the displacement thickness grows parabolically as

$$
\delta^* \propto x^{\frac{1}{2}}
$$

(29)

Differentiating equation (27a) with respect to $x$ yields

$$
\frac{d\delta^*}{dx} = \frac{4}{3} a_1 \left( \frac{M_\infty}{Re} \right)^{\frac{1}{3}} x^{-\frac{1}{3}}
$$

(30a)

which can alternatively be expressed in terms of the weak viscous interaction parameter as

$$
M_\infty \frac{d\delta^*}{dx} = \frac{4}{3} a_1 \bar{x}_w
$$

(30b)
This expression can be substituted in the relationship obtained for the parameter K, equation (11b), to give

$$K = M_{\infty} \frac{dy}{dx} = \frac{4}{3} a_1 \bar{X}_w$$  \hspace{1cm} (31)

This value of K can be used in equation (26a) and the result is

$$P = 1 + \gamma \left( \frac{4}{3} a_1 \bar{X}_w \right) + \gamma \frac{\gamma + 1}{4} \left( \frac{4}{3} a_1 \bar{X}_w \right)^2 + ...$$ \hspace{1cm} (32a)

This expression for air where $\gamma = 1.4$ and adiabatic wall conditions ($T_w = T_o$) becomes

$$P = 1 + 0.062 \bar{X}_w + 0.00164 \bar{X}_w^2 + ...$$ \hspace{1cm} (32b)

which compares well with results given by Barnes and Tang [12]

$$P = 1 + 0.06 \bar{X}_w + 0.00152 \bar{X}_w^2 + ...$$ \hspace{1cm} (32c)

Also, under cold-wall conditions ($T_w = 0$), equation (32a) becomes

$$P = 1 + 0.057 \bar{X}_w + 0.0014 \bar{X}_w^2 + ...$$ \hspace{1cm} (32d)

In a similar way to that for the strong solution, the weak solution can be obtained for flow farther back on a sharp flat plate by solving equations (18) and (30a) simultaneously for a given set of Mach number and wall temperature ratio.
4.1.3 Matching of Strong and Weak Solutions

A complete solution for pressure distribution in hypersonic, turbulent viscous flow past a sharp edge flat plate can be obtained by matching the strong and weak solutions. A comparison of the pressure distribution for a sharp flat plate at zero incidence using the aforementioned method, the pressure law suggested by Barnes and Tang [12] and the results obtained by Stollery and Bates [13] is shown in figure 4 in which \( M_\infty = 9 \), \( T_w = 295 \) K, \( T_0 = 1070 \) K, \( Re = 5.5 \times 10^5/cm \).

The parameter \( a_1 \) is given by equation (19b) and is solely dependent on the wall temperature ratio. The scaled coordinate \( X \) is given by

\[
X = \frac{x}{l}
\]

(33a)

where

\[
l = \frac{M_\infty g C_\infty x}{Re_x}
\]

(33b)

or

\[
X = \frac{Re_x}{M_\infty^g C_\infty}
\]

(33c)

Also, the strong and weak turbulent viscous interction parameters can be written in
Figure 4: Flat Plate Pressure Distribution, $M = 9$, $\alpha = 0$. 
terms of $X$ as

$$\bar{X}_s = \left( \frac{1}{X} \right)^2$$  \hspace{1cm} (34)

and

$$\bar{X}_w = \left( \frac{1}{X} \right)^{1/5}$$  \hspace{1cm} (35)

Equations (34) and (35) reveals that the strong and weak viscous interaction parameters differ merely by a small change in their exponents. Therefore, these choices of the scaling parameters are appropriate for showing the effect of the turbulent viscous interaction parameters and they provide the possibility of extending the computation from very close to the leading edge of a flat plate to very far back. Further, the ratio of $X$ to $a_1^5$ clearly identifies the relationship between the viscous parameter and the wall temperature ratio since $\frac{X_s}{a_1} = \left( \frac{\bar{X}_w}{a_1} \right)^{1/5}$.

It can be seen that close to the leading edge, the pressure has a value much greater than unity which drops rapidly to the value of one further downstream. Also, the results obtained here compare well with the results obtained by Stollery and Bates [13]. However, they employed the simple and more approximate expression for the displacement thickness, equation (4e), while in this study the less approximate and more general form of the displacement thickness, equation (4a), has been used. Barnes and Tang’s results [12] do not admit the solution obtained here or that obtained by Stollery and Bates [13] in the strong viscous interaction region. It is no surprise that the Barnes and Tang’s solution [12]
diverges in the strong viscous interaction region since their pressure law was derived based on the weak interaction theory which is valid only for small values of $\alpha$ and $K$. Also, the series expansions used to derive this pressure law, equation (32c), cannot be expected to hold in this region. It is interesting to note that the adiabatic wall condition was assumed by Barnes and Tang [12] in the derivation of their pressure law while the wall temperature ratio used by Stollery and Bates [13] was $\frac{T_w}{T_o} = 0.275$. In contrast, Barnes and Tang's [12] weak viscous interaction solution compares well with the results obtained by the other methods.

In practice a region of strong turbulent viscous interaction is very unlikely since the transition Reynolds numbers are so high. Therefore, there is no experimental data available for the strong turbulent viscous interaction region to be compared with the strong solution presented here. However, the weak solution for a flat plate is of particular interest in computing the turbulent viscous interaction past a compression or expansion corner. This will be discussed further in the subsequent section.

The method outlined here can be easily extended to the case of a flat plate with incidence. The remaining difference is that for a flat plate with incidence the geometric body shape $y_w$ is not zero. Therefore, the effective body shape can no longer be represented by equation (16a). The correct form of this equation for a flat plate with incidence is

$$y_e(x) = \delta^*(x) + x \tan \alpha \quad (36)$$

in which the geometric body shape is expressed in terms of the distance from the
leading edge of the flat plate and the tangent of the incidence angle.

4.2 Compression Corner

The strong viscous-inviscid interactions which occur at the wing-flap function of control surfaces and on intake compression ramps are of particular interest in the design of any hypersonic vehicle. The behavior of the boundary layer on these control surfaces can be understood by study of a simple compression corner. When an unseparated turbulent boundary layer at hypersonic speeds encounters a two dimensional compression corner, some features of the flow can be predicted using the method described in the following section.

The basic steps in predicting the development of the boundary layer and the pressure distribution for a compression corner follow those of the flat plate. First, it is necessary to determine the geometric body shape and the effective body shape. Then an expression for the displacement thickness is formulated and finally the expressions obtained for the displacement thickness and the pressure are solved simultaneously.

A compression ramp is illustrated in figure 5. The geometric body shape for this corner can be mathematically expressed by the following piecewise continuous functions:

\[ y_w(x) = 0 \quad \text{for} \quad x \leq L \quad (37a) \]
Figure 5: Compression Corner Geometry.
\[ y_w(x) = x \tan \alpha \quad \text{for} \quad x > L \]  \hspace{1cm} (37b)

where \( x \) is the distance from the leading edge of the flat plate, \( L \) is a characteristic length measured from the sharp leading edge of the plate to the hinge line of the corner and \( \alpha \) is the ramp angle. The representation of the geometric body shape by this mathematical model clearly indicates that a compression corner can be viewed as two distinct sections. The first section from the sharp leading edge to the hinge line of the corner can be considered as a flat plate at zero incidence. However, the remaining section, from the hinge line to the end of the corner, can be viewed as a sharp flat plate at incidence. Thus, a full solution of the hypersonic, turbulent viscous flow past a compression corner can be constructed from the solution obtained for these two sections. The solution for the flat plate section can be obtained by employing the method as described in section 4.1 for a given set of initial conditions.

The prediction of the pressure distribution and the displacement thickness over the ramp section can be accomplished through the following procedure. The geometric body shape, equation (37b), for the ramp can be differentiated with respect to \( x \) to yield

\[ \frac{dy_w(x)}{dx} = \tan \alpha \]  \hspace{1cm} (37c)

Then the rate of change of the effective body shape, equation (12b), takes the form
\[ \frac{dy_\varepsilon(x)}{dx} = \frac{d\delta^*}{dx} + \tan \alpha \] (38)

and the parameter K, equation (11b), becomes

\[ K = M_\infty \left( \frac{d\delta^*}{dx} + \tan \alpha \right) \] (39)

This expression can be substituted in the tangent-wedge rule, equation (11a), to yield the following expression:

\[ P = 1 + \gamma M_\infty^2 \left( \frac{d\delta^*}{dx} + \tan \alpha \right)^\gamma \left( \frac{\gamma + 1}{4} \right) + \left( \frac{\gamma + 1}{4} \right)^\gamma + \frac{1}{M_\infty^2 (d\delta^*/dx + \tan \alpha)^2} \right] \] (40)

Now it remains to evaluate the rate of change of the displacement thickness \( \frac{d\delta^*}{dx} \).

The general expression for the displacement thickness, equation (4a), can still be used. As was the case for the flat plate at zero incidence, \( P \) should be specified in this equation before the derivative of the displacement thickness \( \delta^* \) with respect to \( x \) can be obtained. A rather logical way of specifying \( P \) is to take the approach that was employed for the flat plate. In the region close to the hinge line over the ramp the boundary layer can be expected to grow exponentially. An exponential pressure rise was suggested by Oswatitsch and Weighardt [28] and was calculated by a perturbation of the boundary layer by Lighthill [29]. Therefore, \( P \) can be assumed to take the form

\[ P = b \exp(sx) \] (41)
where b and s are constants. This equation can be substituted in the general expression for the displacement thickness, equation (19c) and differentiated with respect to x to yield

$$\frac{d\delta^*(x)}{dx} = a_1 \left( \frac{M_\infty^4 C_\infty}{Re_x} \right)^{\frac{1}{2}} \frac{b^{\frac{1}{2}(\frac{\eta}{2\eta} - \kappa)}}{(sn)^{\frac{1}{2}}} \left( \frac{\frac{4}{3} \eta}{\kappa} - \kappa \right) e^{\frac{sz}{2}(\frac{\eta}{2\eta} - \kappa)}$$  \hspace{1cm} (42)

The constants b and s can be evaluated using the value of pressure at the hinge line obtained previously by matching to the upstream, flat plate solution.

Very far downstream over the ramp the rate of change of the displacement thickness is small. Therefore, P can be assumed to be constant as it was the case for the flat plate at zero incidence. Thus, the rate of growth of the displacement thickness can be computed using equation (30a). Therefore, the pressure distribution further downstream on the ramp can be predicted by solving equations (30a) and (40) simultaneously. The obtained solution for this region can be matched with the solution generated for the region very close to the hinge line over the ramp. This final matching completes the solution of the turbulent, hypersonic flow past a compression corner.

Figure 6 illustrates a comparison of the pressure distribution for turbulent, hypersonic viscous flow past a compression corner obtained by the present study against that of Stollery and Bates [13], with experimental data of Elfstrom [16] and the final inviscid value of the pressure, where $M_\infty = 9.22$, $Re = 5.5 \times 10^5$ /cm, $T_w = 295$ K, $T_o = 1070$ K, $L = 43$ cm, $\alpha = 15$ degrees and the hinge line is at $x = L$. The inviscid pressure distribution was calculated using the inviscid form of the tangent-wedge rule, equation (11c), with plus sign. Figure 6 shows that the
Figure 6: Compression Corner Pressure Distribution, $M = 9.22$, $\alpha = 15^\circ$. 
agreement between the theoretical and experimental data for a 15-degree compression corner is remarkably good, considering the simplicity of the mathematical model and the closeness of the experimental flows to incipient separation. Further this figure reveals that in the region very close to the hinge line over the ramp, the pressure rises drastically and follows an almost straight line which flattens out further downstream. This behavior of the pressure can be best explained in view of the changes of the parameter $K$. As shown before, $K$ is influenced by the combined effect of incidence and displacement thickness. In the region close to the leading edge over the flat plate part of the compression corner, the incidence does not have any effect on $K$ while the effect of the displacement thickness is strong. As the boundary layer encounters the sudden change of incidence close to the hinge line over the ramp, the displacement thickness thins dramatically in the adverse pressure gradient which in turn causes the pressure to rise very rapidly. Further downstream on the ramp, the displacement thickness grows thicker again, the effect of incidence becomes predominant and the pressure flattens out and exceeds slightly the final inviscid two-dimensional. The reason for this slight excess pressure is that the effective body shape $y_e(x)$ has a greater slope than the corner angle. It can be concluded that for hypersonic flow at a given Mach number past a flat plate or a ramp with a given corner angle, $K$ is a function of one independent variable, namely the rate of change of the displacement thickness. Also, the tangent-wedge rule for pressure is a function of $K$ only. Therefore, the pressure is directly influenced by variations of the rate of change of the displacement thickness.
4.3 Expansion Corner

The interaction between expansions and turbulent boundary layers is a problem which has received little attention. This study aims to add some insights to such an interaction.

4.3.1 Method of Solution

Figure 7 illustrates a schematic of an expansion corner. Comparison of this figure with the compression corner, figure 5, shows that the difference between these two cases is the turning angle of the flow. The turning angle for the compression corner was taken as positive while the flow turning angle is negative for the expansion corner. Thus, the mathematical model describing the expansion corner is the same as the compression corner with the angle being negative in the former case. Therefore, the geometric body shape can be expressed in terms of the following piecewise continuous functions:

\[
y_w(x) = 0 \quad \text{for} \quad x \leq L \quad \Delta a
\]

\[
y_w(x) = -x \tan \alpha \quad \text{for} \quad x > L \quad \Delta b
\]

The similarity between the mathematical models for the compression and the expansion corners suggests the possibility of treating both flows with the same approach but with different boundary conditions. The expansion corner can be modeled as a sharp flat plate at zero incidence followed by a flat plate with negative incidence. The solution for the hypersonic, turbulent flow past the sharp
Figure 7: Expansion Corner Geometry.
flat plate section is already known from section 4.1 while the geometric body shape \( y_w(x) \) for the expansion corner, equation (43b), can be differentiated with respect to \( x \) to yield

\[
\frac{dy_w(x)}{dx} = -\tan\alpha \tag{43c}
\]

The above equation can be substituted in the relationship for the rate of change of the effective body shape, equation (12b), to yield

\[
\frac{dy_e(x)}{dx} = \frac{d\delta^*}{dx} - \tan\alpha \tag{44}
\]

The parameter \( K \), equation (11b), becomes

\[
K = M_\infty (\frac{d\delta^*}{dx} - \tan\alpha). \tag{45}
\]

The appropriate form of the rate of change of the displacement thickness must then be formed. This is the subject of the next section.

4.3.2 Displacement Thickness for Expansion Corner

The general expression for the displacement thickness obtained from the momentum-integral equation was shown in equation (4a). Also, the initially unknown pressure \( P \) in this expression should be specified before equation (4a) can be differentiated with respect to \( x \) to yield an expression for the rate of change of the displacement thickness. In the strong viscous interaction region of a sharp flat
plate at zero incidence, $P$ could take a power law form while in the weak viscous interaction region of a flat plate $P$ could be assumed to be constant. For a compression corner, an exponential assumption turned out to be appropriate. Hence, it is necessary to conduct an investigation for the appropriate form of $P$ for an expansion corner flow. Three different forms of $P$ have been considered. These forms are logarithmic, power law and exponential. The logarithmic form was disregarded due to the singularity of the logarithmic function at zero and infinity. Also, if $P$ is assumed to take the form $P = \ln x^Q$, where $x$ is the distance from the leading edge of the corner as shown in figure 7, then at a point downstream where $x$ becomes equal to unity the pressure suddenly becomes zero which is not physically possible. The power and exponential laws seem to show some validity. Nevertheless, further distinction regarding the validity of these assumptions is not possible until the problem is solved completely. Appeal must be made to experiments to determine which approximation is more appropriate. This means that these expressions should be substituted in the relationship obtained for the displacement thickness, equation (4a), this relationship should be differentiated with respect to $x$ to yield the rate of change of the displacement thickness. Finally the rate of change of the displacement thickness and an appropriate form of the pressure law should be solved simultaneously to reach the solution. Comparison of the obtained result by using these assumptions with the experimental data will provide a way of judging the validity of these assumptions. Choice of the pressure law and the results of this investigation are the subject of the subsequent sections.
4.3.3 Pressure Law for Expansion Corner

The importance of the choice of the pressure law in solving hypersonic flows involving the joint effects of the viscous interaction and the incidence was shown by Mohammadian [8] and Stollery [11]. The results obtained for the flat plate at zero incidence and the compression corner in the current study suggest that the tangent-wedge rule approximation for the pressure gives satisfactory results. However, the tangent-wedge approximation in the form of equation (11a) cannot be used to predict the pressure distribution over an expansion corner. The reason is the right hand side of this equation is always greater than or equal to unity regardless of the value of K. One way of accounting for the actual pressure drop past an expansion is perhaps to change the plus sign to a minus sign in the tangent-wedge rule approximation, equation (11a), to yield

$$P = 1 + \gamma K^2 \left( \frac{\gamma + 1}{4} - \left( \frac{\gamma + 1}{4} \right)^2 + \frac{1}{K^2} \right)$$

This form of the tangent-wedge rule seems to be appropriate for treatment of expansion corners since it clearly indicates that the pressure will always have a value less than unity beyond the hinge line. In fact, this is very much like equation (11c) with a minus sign which was suggested by Stollery [11] for calculation of the final inviscid two-dimensional value of the pressure over an expansion corner. The only difference is that K in equation (46), replaces Ma in equation (11c). Although at first glance equation (46) appears to be suitable for expansion corners, the presence of K creates conditions that require further study of this equation. Recall that K is affected by the rate of change of displacement
thickness and incidence. Also, it was mentioned in the previous section that the boundary layer is thick close to the hinge line. In the region very close to the hinge line of the expansion corner, this thick boundary layer experiences an abrupt drop in incidence. This sudden drop of incidence causes the thick boundary layer to thicken even further in the region close to the hinge line over the ramp section of the expansion corner. This is opposite to the compression corner where the positive change of incidence forces the boundary layer to thin further downstream. For the expansion corner, the thickening of the boundary layer causes $K$ to rise just downstream of the hinge line. Therefore, $K$ becomes very large and the value of the pressure calculated from equation (46) underpredicts the real value of the pressure just downstream of the hinge line. Clearly, further downstream over the ramp where the growth of the boundary layer is smoother the value of the pressure calculated from equation (46) gives a better agreement with experimental data. Thus, this form of the tangent-wedge rule is not suitable for the complete prediction of the pressure distribution over an expansion corner.

It is suggested that the following pressure laws for the prediction of the pressure distribution for turbulent flow past expansion corners at hypersonic and supersonic speeds be used:

\[
P = 1 + AK + BK^2 + \bar{v}_w^{-0.4}(M\alpha) \quad \text{for} \quad M\alpha \leq 1 \quad (48a)
\]

\[
P = 1 + AK + BK^2 + \bar{v}_w^{0.4}(M\alpha) \quad \text{for} \quad M\alpha > 1 \quad (48b)
\]

where
\[ A = \gamma \]  
\[ B = \gamma \frac{\gamma + 1}{2} \]  

and \( \chi_w \) is the weak viscous interaction parameter given by equation (15). These are second-order equations in \( K \) where the coefficients A and B are purely dependent on the ratio of specific heats \( \gamma \). The last term in these equations describes the interaction between the weak viscous interaction parameter and the incidence where the exponents are empirical constants. Note that both equations (48a) and (48b) at zero incidence where \( \alpha \) and \( K \) are zero give a value of the pressure equal to unity. This compares very well with the experimental value of the pressure close to the hinge line over the flat plate section of a corner. The appearance of the strong and weak approximations to the tangent-wedge rule, equations (22b) and (26b), in these relationships confirm their validity for the small and large values of \( K \). Thus, the last term in these equations can be viewed as a correction for the joint effects of viscosity and incidence. It can be seen that for the cases where the Mach number and the incidence are large the necessary correction is considerably larger compared to the case where \( M\alpha \) is less than unity.

4.3.4 Results and Discussion for Expansion Corner

Section 4.3.2 was concluded by finalizing the choices of the initially unknown pressure \( P \) in the expression for the displacement thickness to two choices of power law and exponential forms. Also, it was mentioned that the final
decision must be postponed until the appropriate form of the pressure law is achieved and the flow is solved. The approach is to solve the flow past an expansion corner, check the results versus the experimental data and select the form for $P$ which gives the closest agreement for the displacement thickness and the pressure distribution. The solution over the flat plate section of the expansion corner is currently described in section 4.3.1. The flow properties past the ramp section of the expansion corner can be predicted by simultaneously solving equations (21b) or (42) depending on the assumption for $P$ in the displacement thickness relationship, equation (45) and the pressure law based on the value of $Ma$. The complete solution for the flow past an expansion corner can be formulated by matching the value of the pressure at the hinge line obtained from the solution of the flat plate section and the solution of the ramp section. As was the case for the compression corner, the matching of the upstream and downstream values of the pressure at the hinge line provides values of the constants in equations (20) or (41) whichever is employed.

Figure 8 illustrates a comparison of the results obtained for hypersonic, turbulent viscous flow past a 2.5-degree expansion corner. Expressed in the figure are the results of the displacement thickness relationship with the power law assumption and two different pressure laws against the case where the displacement thickness relationship with exponential assumption, and the suggested pressure law are used versus the experimental data given by Lu and Chung [18]. The pressures are plotted in non-dimensional form against $\bar{x} = \frac{x}{\delta_0}$ where $x$ is the surface coordinate from the corner. This graph shows that once the rate of growth of the displacement thickness is predicted based on the power law
Figure 8: Expansion Corner Pressure Distribution, $M = 8$, $\alpha = 2.5^\circ$. 
assumption, neither one of the pressure laws are capable of predicting the pressure distribution accurately. However, the most satisfactory results are achieved when the rate of change of the displacement thickness is calculated assuming that the initially unknown pressure $P$ in this relationship takes an exponential form, equation (42) and the suggested pressure law, equation (48a), is employed. Note that all graphs show correct asymptotic behavior toward the final inviscid value of the pressure downstream of the expansion corner. The initial conditions for figure 8 were identical to those of Lu and Chung [18] where $T_w = 290 ~K$, $T_o = 820 ~K$, $M_{\infty} = 8$, $Re = 1.02 \times 10^5/cm$. Also, to further secure the validity of the above claim regarding the power law and exponential assumptions, the calculation was performed for several different cases. However, due to the similarity of the results and for brevity only the results for the 2.5-degree expansion case is shown.

The behavior of the geometric body shape $y_{w}(x)$, the effective body shape $y_{e}(x)$ and the displacement thickness $\delta^*$ for the ramp section of the 2.5-degree expansion corner are displayed in figure 9. The convex geometric body shape is merely a straight line with negative slope. The displacement thickness and the effective body shape have the same value at the hinge line. Close to the hinge line over the ramp section where the effect of the displacement thickness dominates the effect of the incidence, the absolute values of the displacement thickness and the effective body shape are very close. As the flow moves downstream the effect of the displacement thickness reduces and the effect of incidence becomes more dominant. Therefore, the effective body shape deviates from the displacement thickness. This behavior continues further downstream and at $x = 5.45 ~cm$ the positive displacement thickness and the negative
Figure 9: Boundary Layer Properties, $M = 8$, $\alpha = 2.5^\circ$. 
geometric body shape become equal so that the effective body shape becomes zero. This point can be viewed as the border between the region of strong and weak interaction. The division of the interaction region in this manner is comparable to the strong and weak viscous interaction regions over a flat plate at zero incidence as described in chapter 3 of this study. Further downstream, the displacement thickness continues to drop which causes the effective body shape to become negative toward the end of the corner.

The rate of growth of the geometric body shape, the displacement thickness and the effective body shape for the ramp section of the 2.5 degree expansion corner are compared in figure 10. The rate of change of the displacement thickness and the geometric body shape are negative just after the hinge line. The sum of these two negative numbers results in a large negative value of the rate of change of the effective body shape in this region. Since the rate of change of the effective body shape is large, consequently, the magnitude of the parameter K is also large. This large value of K is the cause for the very rapid drop of the pressure in the region just downstream of the hinge line. The rate of change of the displacement thickness increases moderately as the flow moves further downstream which contributes to a slow increase of the rate of change of the effective body shape and K. This weak change of K explains the asymptotic behavior of the pressure toward the end of the ramp.

Now that the effectiveness of the method by using the exponential assumption in the rate of change of the displacement thickness relationship is tested against experimental data and the results are satisfactory, it seems reasonable to continue the treatment of the other expansion corners with the same
Figure 10: Rate of Change of Boundary Layer Properties, $M = 8, \alpha = 2.5^\circ$. 
Figure 11: Expansion Corner Pressure Distribution, $M = 8$, $\alpha = 4.25^\circ$. 
approach. The pressure distribution over a 4.5-degree expansion corner obtained by the present method is compared against the experimental data given by Lu and Chung [18] in figure 11. The initial conditions for this case are the same as those of the 2.5-degree expansion corner presented earlier. The agreement between the theoretical and the experimental data is remarkably good. As expected the calculated pressure tends asymptotically to the inviscid two-dimensional value of the pressure further downstream.

Figure 12 illustrates a comparison of the results obtained for the hypersonic, turbulent viscous flow past 5, 10 and 15 degrees expansion corners using the present theory against Bloy's experimental data [17], with the following conditions: $T_w = 280$ K, $T_o = 2160$ K, $M_\infty = 7.35$, $Re = 2.9 \times 10^5$/cm. All these cases indicate a good agreement of the theory with the experiment. The pressure levels tend asymptotically to their inviscid limits toward the end of the corner.

An investigation of the possibility of extending the current method to supersonic turbulent flows past expansion corners has been performed by testing the method against some experimental data for supersonic cases. A comparison with Goldfeld's experimental data [19] at Mach 3 and 4 past a 15-degree expansion corner is illustrated in figure 13, where $\frac{T_w}{T_o} = 1$ and $Re = 2.2 \times 10^4$/cm. The theory agrees with the experiment very well. As was the case for hypersonic turbulent flows, the pressure distribution for supersonic flows tends asymptotically toward the inviscid value further downstream toward the end of the expansion corners. The results obtained for supersonic cases prove the applicability of the method for supersonic, turbulent viscous flow past expansion corners.
Figure 12: Expansion Corner Pressure Distribution $M = 7.5$. 
Figure 13: Expansion Corner Pressure Distribution, $\alpha = 15^\circ$. 

- Present theory
- Goldfeld, $M = 3$
- Goldfeld, $M = 4$
- Inviscid
The inviscid values of the pressure were calculated using the inviscid form of the tangent-wedge rule with the minus sign, equation (11c), for all hypersonic and supersonic flows past expansion corners presented in this study within the range of validity of the inviscid tangent-wedge rule. The limit of the inviscid tangent-wedge rule for explosive flows as explained in section 2.2.2 of the current study, is set by $Ma = -1.89$ which corresponds to $P_f = 0$. Thus, for the flow at Mach 7.35 past 15-degree expansion corner where $Ma = -1.924$ the inviscid tangent-wedge rule is not applicable. Therefore, a Prandtl-Meyer expansion allowed the asymptotic value of the pressure to be obtained for this case.
5. CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The turbulent viscous interaction problem at hypersonic speeds on a flat plate, a compression corner and an expansion corner was solved in a similar way to that described by Stollery [13]. The tangent-wedge approximation used for the prediction of the pressure distribution past flat plate and compression corner showed satisfactory results. The expansion corner was treated with an appropriate pressure law approximation. The proposed pressure law showed explicit dependence on the hypersonic viscous interaction parameter. The growth of the boundary layer displacement thickness in an unknown pressure gradient for flow at any given Mach number and wall temperature ratio over an expansion corner was obtained by using the momentum-integral equation and assuming that the initially unknown pressure showed an exponential decay. The effect of the wall temperature ratio on the displacement thickness ratio was best shown once a representative value of $n = 8.5$ in the latter relationship was used. The method utilized for the hypersonic flow was then applied to turbulent supersonic flow past an expansion corner to further substantiate the capability of the theory. Comparison of the results with the experimental data demonstrated promising agreement. Considering the simplicity of the analysis involved in this study the results are remarkably good. This method provides a simple and quick tool for prediction of important flow properties and can be used in preliminary design of the hypersonic vehicles. The method is unable to predict the upstream

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propagation, which is insignificant in supersonic and hypersonic turbulent flows past expansion corners.

5.2 Recommendations

The validity of the exponential assumption for the initially unknown pressure in the displacement thickness relationship for flow past a flat plate and other two-dimensional shapes needs to be investigated. This assumption needs to be tested in the relationship given by Stollery [13] for the prediction of the Stanton number and the skin-friction coefficient since the Stanton number is also a function of the initially unknown pressure. The fact that the effective body shape becomes zero at a certain location past an expansion corner can be investigated further for several different cases that may lead to a trend or criteria for the zone of influence of the strong and weak viscous interactions. It would, moreover, be very helpful to investigate the problem using, for example, triple-deck theory to study in more detail the turbulent expansion corner flow problem.
APPENDIX

DERIVATION OF THE DISPLACEMENT THICKNESS RATIO
Recall that the Crocco [23] velocity-temperature relationship can be written as:

\[
\frac{T}{T_e} = \frac{T_w}{T_e} + \left( \frac{T_r - T_w}{T_e} \right) \frac{u}{u_e} - \left( \frac{T_r - T_e}{T_e} \right) \left( \frac{u}{u_e} \right)^2
\]  
(A-1)

or by taking

\[
a = \frac{T_w}{T_e}
\]  
(A-2)

\[
b = \left( \frac{T_r - T_w}{T_e} \right)
\]  
(A-3)

\[
c = \left( \frac{T_r - T_e}{T_e} \right)
\]  
(A-4)

can be written as

\[
\frac{T}{T_e} = a + b \left( \frac{u}{u_e} \right) - c \left( \frac{u}{u_e} \right)^2
\]  
(A-5)

Also, the boundary layer thickness \( \delta \) can be written as:

\[
\delta = \int_0^\delta dy = \int_0^\Delta \frac{\rho \Delta}{p'} dY = \int_0^\Delta \frac{T}{T_e} dY
\]  
(A-6)

and the velocity profile with power law dependence can be written as:

\[
\frac{u}{u_e} = \left( \frac{Y}{\Delta} \right)^{\frac{1}{k}}
\]  
(A-7)
Substituting equation (A-7) in equation (A-5) and equation (A-5) in equation (A-6) results in

\[ \delta = \Delta \int_0^1 \left\{ a + b \left( \frac{Y}{\Delta} \right)^{\frac{n}{n+1}} - c \left( \frac{Y}{\Delta} \right)^{\frac{n}{n+2}} \right\} d\left( \frac{Y}{\Delta} \right) \]  
(A-8)

This can be integrated to give:

\[ \delta = \Delta \left\{ a + b \left( \frac{n}{n+1} \right) - c \left( \frac{n}{n+2} \right) \right\} \]  
(A-9)

Recall also that the displacement thickness ratio can be shown to be:

\[ \frac{\delta^*}{\delta} = \frac{\delta - \Delta J}{\delta} = 1 - \frac{\Delta J}{\delta} \] 
(A-10)

where

\[ J = \frac{n}{n+1} \] 
(A-11)

Substituting equations (A-9) and (A-11) in equation (A-10) results in

\[ \frac{\delta^*}{\delta} = 1 - \frac{n}{a + b \left( \frac{n}{n+1} \right) - c \left( \frac{n}{n+2} \right)} \] 
(A-12)

Replacing a, b and c with their corresponding relationships given in equations (A-2)-(A-4) gives:
\[
\frac{\delta^*}{\delta} = 1 - \frac{n}{n + 1} \left( \frac{T_w}{T_e} + \left( \frac{T_r - T_w}{T_e} \right) \left( \frac{n}{n + 1} \right) \right) - \left( \frac{T_r - T_e}{T_e} \right) \left( \frac{n}{n + 2} \right)
\] (A-13)

which can be simplified to the final expression for the displacement thickness ratio:

\[
\frac{\delta^*}{\delta} = 1 - \frac{n + 2}{\left( \frac{T_r}{T_e} \right) + \left( \frac{n + 2}{n} \right) \left( \frac{T_w}{T_e} \right) + (n + 1)}
\] (A-14)
REFERENCES


